

WEAK SOLUTIONS OF THE CHERN-RICCI FLOW ON COMPACT COMPLEX SURFACES

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ABSTRACT. In this note, we prove the existence of weak solutions of the Chern-Ricci flow through blow downs of exceptional curves, as well as backwards smooth convergence away from the exceptional curves on compact complex surfaces. The smoothing property for the Chern-Ricci flow is also obtained on compact Hermitian manifolds of dimension n under a mild assumption.

1. INTRODUCTION

Let (M, g_0) be a compact Hermitian manifold with associated $(1, 1)$ form ω_0 . The Chern-Ricci flow starting at ω_0 is given by

$$\frac{\partial}{\partial t}\omega = -\text{Ric}(\omega), \quad \omega|_{t=0} = \omega_0, \quad (1.1)$$

where $\text{Ric}(\omega) = -\sqrt{-1}\partial\bar{\partial}\log\det g$ is the Chern-Ricci form of ω . It was introduced by Gill [5] on manifolds with vanishing first Bott-Chern class and investigated by Tosatti and Weinkove [20] in details on general Hermitian manifolds. If the initial metric is Kähler, then it coincides with the Kähler-Ricci flow.

Many nice properties of the flow have been found (see [3, 20–22], etc.), some of which are analogous to those of the Kähler-Ricci flow. Let

$$T = \sup\{t \geq 0 \mid \exists \psi \in C^\infty(M), \omega_0 - t\text{Ric}(\omega_0) + \sqrt{-1}\partial\bar{\partial}\psi > 0\}.$$

It was proved in [20] that there exists a unique maximal solution $\omega(t)$ to the Chern-Ricci flow (1.1) on $[0, T)$. It is expected that the Chern-Ricci flow is closely related to the geometry of the underlying manifold. In the case of $n = 2$ (complex surfaces), the behavior of the flow is particularly interesting. Let M be a compact complex surface with ω_0 a Gauduchon metric. It was proved in [20] that the Chern-Ricci flow starting at ω_0 exists until either the volume of M goes to zero, or the volume of a curve of negative self-intersection goes to zero. The Chern-Ricci flow is said to be *collapsing* (*non-collapsing*) at T if the volume of M with respect to $\omega(t)$ goes to zero (stays positive) as $t \rightarrow T^-$.

Suppose that the Chern-Ricci flow starting at ω_0 is non-collapsing at $T < \infty$. It was shown in [20] that M contains finitely many disjoint (-1) -curves E_1, \dots, E_k and thus there exists a map $\pi : M \rightarrow N$ onto a complex surface N contracting each E_i to a point $y_i \in N$. It was conjectured in [21] that the flow blows down the exceptional curves and continues in a unique way on a new surface N . The conjecture requires

smooth convergence of the metrics away from the (-1) -curves and global Gromov-Hausdorff convergence as $t \rightarrow T^-$ and $t \rightarrow T^+$. Denote $M' = M \setminus \cup_{i=1}^k E_i$. In [21], Tosatti and Weinkove prove the following theorem.

Theorem 1.1. (Tosatti-Weinkove) *With the notation above, then the metrics $\omega(t)$ converge to a smooth Gauduchon metric ω_T on M' in $C_{\text{loc}}^\infty(M')$ as $t \rightarrow T^-$. Assume in addition*

$$\omega_0 - T \text{Ric}(\omega_0) + \sqrt{-1} \partial \bar{\partial} f = \pi^* \omega_N \quad (1.2)$$

for some $f \in C^\infty(M, \mathbb{R})$ and ω_N a smooth $(1,1)$ form on N . Then there exists a distance function d_T on N such that (N, d_T) is a compact metric space and $(M, g(t)) \rightarrow (N, d_T)$ as $t \rightarrow T^-$ in the Gromov-Hausdorff sense.

Denote $\tilde{\omega}_T = (\pi^{-1})^* \omega_T$ the push-down of the limiting current ω_T to N . To continue the flow on N , first we need to show that the Chern-Ricci flow on N with singular initial metric $\tilde{\omega}_T$ has a unique smooth solution on $(T, T']$ for some $T' > T$. To prove the smooth convergence on compact subsets of $N' = N \setminus \{y_1, \dots, y_k\}$, we need more precise estimates near the exceptional curves as $t \rightarrow T^-$ and also estimates on $[T, T'] \times N'$. We may assume ω_N in condition (1.2) to be a Gauduchon metric after replacing f by a new function (see Lemma 3.2 in [21]) and denote

$$T_N = \sup\{t \geq T \mid \exists \psi \in C^\infty(M), \omega_N - (t - T) \text{Ric}(\omega_N) + \sqrt{-1} \partial \bar{\partial} \psi > 0\}.$$

Using the construction of Song-Weinkove for the Kähler-Ricci flow in [14], we prove the following theorem.

Theorem 1.2. *Assume that the condition (1.2) is satisfied. With the notation above, there exists a unique maximal solution $\omega(t)$ to the equation:*

$$\omega|_{t=T} = \tilde{\omega}_T, \quad \frac{\partial}{\partial t} \omega = -\text{Ric}(\omega), \quad \text{for } t \in (T, T_N),$$

which is smooth on (T, T_N) . Moreover, $\omega(t)$ converges to $\tilde{\omega}_T$ in $C_{\text{loc}}^\infty(N')$ as $t \rightarrow T^+$.

When ω_0 is Kähler, the result is contained in the work of Song-Weinkove [14]. We use the techniques of Song-Weinkove [14], Tosatti-Weinkove [20, 21] and a trick of Phong-Sturm [11] in the proof of the above theorem.

We actually obtain an existence theorem on M of general dimension n . Let Ω be a smooth volume form on M and $\hat{\omega}_t = \omega_0 + t\chi$, where χ is a closed $(1,1)$ form locally defined by $\chi = \sqrt{-1} \partial \bar{\partial} \log \Omega$. Denote $PSH(M, \omega_0)$ the set of ω_0 -plurisubharmonic functions and let

$$PSH_p(\omega_0, \Omega) = \{\varphi \in PSH(M, \omega_0) \cap L^\infty(M) \mid \frac{(\omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi)^n}{\Omega} \in L^p(M)\}.$$

Following the arguments of Song-Tian in [13], we prove the smoothing property for the Chern-Ricci flow.

Theorem 1.3. Suppose that $\omega'_0 = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_0$ for some $\varphi_0 \in PSH_p(\omega_0, \Omega)$, $p > 1$. Assume that ω_0 satisfies

$$\forall u \in PSH(M, \omega_0) \cap L^\infty(M), \quad \int_M (\omega_0 + \sqrt{-1}\partial\bar{\partial}u)^n = \int_M \omega_0^n, \quad (1.3)$$

then there exists a unique family of smooth metrics $\omega(t)$ on $(0, T)$ such that

- (i) $\frac{\partial}{\partial t}\omega = -\text{Ric}(\omega)$, for $t \in (0, T)$.
- (ii) There exists $\varphi \in C^0([0, T] \times M) \cap C^\infty((0, T) \times M)$ such that $\omega = \hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\varphi$ and $\varphi(t) \rightarrow \varphi_0$ in $L^\infty(M)$ as $t \rightarrow 0^+$.

In particular, $\omega(t) \rightarrow \omega'_0$ in the sense of currents as $t \rightarrow 0^+$.

When (M, ω_0) is a Kähler manifold, the result is contained in the work of Song and Tian [13] (see also [1]). When M is a compact complex surface with a Gauduchon metric ω_0 , condition (1.3) is satisfied and the above result follows. The proof of Theorem 1.3 is given in section 5.

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After this note was completed, the author learned that related results were proved by Tat Dat Tô in [17]. The author would like to thank Tat Dat Tô for sending his preprint.

2. CONSTRUCTION OF THE WEAK SOLUTION

In this section, we construct the solution in Theorem 1.2 explicitly. We will follow the construction of Song-Weinkove for the Kähler-Ricci flow [14].

Without loss of generality, assume that M contains only one exceptional curve E for simplicity. Assuming that the condition (1.2) holds for a Gauduchon metric ω_N on N . Define

$$\hat{\omega}_t = (1 - \frac{t}{T})\omega_0 + \frac{t}{T}\pi^*\omega_N,$$

which are smooth nonnegative forms on $[0, T]$. Let $\Omega = e^{f/T}\omega_0^n$. (From now on, we will write n instead of 2 whenever our calculations hold for $n \geq 2$.) If φ solves the parabolic complex Monge-Ampère equation

$$\frac{\partial \varphi}{\partial t} = \log \frac{(\hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\varphi)^n}{\Omega}, \quad \varphi|_{t=0} = 0 \quad (2.1)$$

for $t < T$, then $\omega = \hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\varphi$ solves the Chern-Ricci flow (1.1) on $[0, T)$.

By Lemma 3.3 of [21], there exists a uniform constant C such that $|\varphi| \leq C$ and $\dot{\varphi} \leq C$, where we write $\dot{\varphi}$ for $\frac{\partial \varphi}{\partial t}$. Then it follows that as $t \rightarrow T^-$, $\varphi(t)$ converges

pointwise on M to a bounded function φ_T with

$$\omega_T = \hat{\omega}_T + \sqrt{-1}\partial\bar{\partial}\varphi_T \geq 0.$$

In particular, $\omega(t) \rightarrow \omega_T$ in the sense of currents as $t \rightarrow T^-$. From Lemma 5.1 of [14] φ_T must be constant on E since $\sqrt{-1}\partial\bar{\partial}\varphi_T|_E = \omega_T|_E \geq 0$. Thus $\psi_T = (\pi^{-1})^*\varphi_T$ is a bounded function on N which is smooth on $N \setminus \{y_0\}$ as π is the blow down map contracting E to y_0 . Then

$$\tilde{\omega}_T = \omega_N + \sqrt{-1}\partial\bar{\partial}\psi_T \geq 0. \quad (2.2)$$

Lemma 2.1. *There exists $p > 1$ such that $\tilde{\omega}_T^n/\omega_N^n \in L^p(N)$. Moreover, ψ_T is continuous on N .*

Proof. The argument of Lemma 5.2 in [14] shows that $\tilde{\omega}_T^n/\omega_N^n \in L^p(N)$. The continuity of ψ_T then follows from the results in [8] and Dinew-Kołodziej [2]. \square

Given a smooth volume form Ω_N , now we will construct a family of functions $\psi_{T,\epsilon}$ on N which converge to ψ_T in $L^\infty(N)$. For sufficiently small $\epsilon > 0$ and K large enough, define

$$\Omega_\epsilon = (\pi|_{M \setminus E}^{-1})^* \left(\frac{|s|_h^{2K} \omega^n(T - \epsilon)}{\epsilon + |s|_h^{2K}} \right) + \epsilon \Omega_N \text{ on } N \setminus \{y_0\}.$$

and $\Omega_\epsilon|_{y_0} = \epsilon \Omega_N|_{y_0}$. Here s is a holomorphic section of the line bundle $[E]$ vanishing along the exceptional curve E to order 1. Choose h to be a smooth Hermitian metric on $[E]$ as in [20] with curvature $R_h = -\sqrt{-1}\partial\bar{\partial} \log |s|_h^2$ such that for sufficiently small $\epsilon > 0$, $\pi^*\omega_N - \epsilon R_h > 0$. (see [6] for an argument of this.) Then the volume form $\Omega_\epsilon \in C^k(N)$ for a fixed constant k as π is the blow down map and K can be chosen to be sufficiently large. Moreover, Ω_ϵ converges to $\tilde{\omega}_T^n$ in C^∞ on compact subsets of $N \setminus \{y_0\}$ as ϵ goes to zero. By the result of Tosatti and Weinkove [19], there exist functions $\psi_{T,\epsilon} \in C^k(N) \cap C^\infty(N \setminus \{y_0\})$ such that

$$(\omega_N + \sqrt{-1}\partial\bar{\partial}\psi_{T,\epsilon})^n = C_\epsilon \Omega_\epsilon$$

where the constants C_ϵ are chosen so that the integrals of both sides of the above equation match. Write $F_\epsilon = C_\epsilon \Omega_\epsilon / \Omega_N$ and $F = \tilde{\omega}_T^n / \Omega_N$. Note that $C_\epsilon \rightarrow 1$ as $\epsilon \rightarrow 0$, then by the definition of Ω_ϵ we have

$$\lim_{\epsilon \rightarrow 0} \|F_\epsilon - F\|_{L^1(N)} = 0.$$

By Lemma 2.1 and Kolodziej's stability result [8], we have

$$\lim_{\epsilon \rightarrow 0} \|\psi_{T,\epsilon} - \psi_T\|_{L^\infty(N)} = 0. \quad (2.3)$$

Let $\chi = \sqrt{-1}\partial\bar{\partial} \log \Omega_N$, then there exists $T' > T$ such that

$$\hat{\omega}_{t,N} = \omega_N + (t - T)\chi$$

are smooth Gauduchon metrics for $t \in [T, T']$. For convenience, we will still write $\hat{\omega}_t$ for $\hat{\omega}_{t,N}$. Consider the parabolic complex Monge-Ampère equations

$$\frac{\partial \varphi_\epsilon}{\partial t} = \log \frac{(\hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\varphi_\epsilon)^n}{\Omega_N}, \quad \varphi_\epsilon|_{t=T} = \psi_{T,\epsilon}$$

on $[T, T']$. Then Theorem 1.3 and Lemma 2.1 give the following proposition.

Proposition 2.1. *There exists $\varphi \in C^0([T, T'] \times N) \cap C^\infty((T, T'] \times N)$ such that*

(i) $\varphi_\epsilon \rightarrow \varphi$ in $L^\infty([T, T'] \times N)$.

(ii) $\omega(t) = \hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\varphi$ is the unique solution to the equation,

$$\omega|_{t=T} = \tilde{\omega}_T, \quad \frac{\partial}{\partial t}\omega = -\text{Ric}(\omega), \quad \text{for } t \in (T, T'].$$

To prove smooth convergence of $\omega(t)$ on compact subsets of $N \setminus \{y_0\}$ as $T \rightarrow T^+$, we need uniform estimates for $\omega_\epsilon(t) = \hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\varphi_\epsilon$ on $[T, T'] \times N \setminus \{y_0\}$, independent of ϵ . To obtain these estimates, we need the bounds at time T , i.e. the bounds for $\tilde{\omega}_{T,\epsilon} = \omega_N + \sqrt{-1}\partial\bar{\partial}\psi_{T,\epsilon}$. We will prove these in section 4.

3. HIGHER ORDER ESTIMATES AS $t \rightarrow T^-$

In this section, we will prove a third order estimate and a bound for $|\text{Ric}|$, which will be used to obtain the bounds for $\tilde{\omega}_{T,\epsilon}$ in the next section.

As $t \rightarrow T^-$, the smooth convergence of $\omega(t)$ on compact subsets of $M \setminus \{E\}$ follows from Theorem 1.1 of [21]. In particular, C^∞ a priori estimates for $\omega(t)$ on compact subsets away from the exceptional curves have already been obtained. However, we need more precise higher order global estimates on M as $t \rightarrow T^-$ to obtain the bounds for $\tilde{\omega}_{T,\epsilon}$.

By Lemma 2.3 and 2.4 in [21], there exist positive constants C and K such that

$$\frac{|s|_h^{2K}}{C}\omega_0 \leq \omega(t) \leq \frac{C}{|s|_h^{2K}}\omega_0. \quad (3.1)$$

We may assume that $|s|_h^2 \leq 1$ on M for convenience. By Lemma 2.1 of Guan-Li [7], we can choose local coordinates around a point such that at this point

$$(g_0)_{i\bar{j}} = \delta_{ij}, \quad \partial_i(g_0)_{j\bar{j}} = 0 \quad (3.2)$$

for all i, j and $(g_{i\bar{j}})$ is diagonal. Now choose such a coordinate system around a point. Following an argument in [18], we can get the inequality,

$$\frac{|\nabla \text{tr}_{\omega_0} \omega|_g^2}{\text{tr}_{\omega_0} \omega} \leq \sum_{i,j,k} g^{i\bar{i}} g^{j\bar{j}} \partial_k g_{i\bar{j}} \partial_{\bar{k}} g_{j\bar{i}} + \sum_{i,j} g^{i\bar{i}} g^{i\bar{i}} |\partial_j(g_0)_{i\bar{j}}|^2. \quad (3.3)$$

To see this, first applying the Cauchy-Schwarz inequality

$$\begin{aligned}
|\nabla \operatorname{tr}_{\omega_0} \omega|_g^2 &= \sum_i g^{i\bar{i}} \left(\sum_j \partial_i g_{j\bar{j}} \right) \left(\sum_k \partial_i g_{k\bar{k}} \right) \\
&= \sum_i g^{i\bar{i}} \left| \sum_j \partial_i g_{j\bar{j}} \right|^2 \\
&\leq \sum_i g^{i\bar{i}} \left(\sum_j g_{j\bar{j}} \right) \left(\sum_j g^{j\bar{j}} |\partial_i g_{j\bar{j}}|^2 \right) \\
&= (\operatorname{tr}_{\omega_0} \omega) \left(\sum_{i,j} g^{i\bar{i}} g^{j\bar{j}} \partial_i g_{j\bar{j}} \partial_{\bar{i}} g_{j\bar{j}} \right). \tag{3.4}
\end{aligned}$$

Note that $\omega = \omega_0 + \theta(t)$ for a closed (1,1) form $\theta(t)$. Hence

$$\partial_i g_{j\bar{j}} - \partial_j g_{i\bar{j}} = \partial_i (g_0)_{j\bar{j}} - \partial_j (g_0)_{i\bar{j}}.$$

Using (3.2), we get

$$\partial_i g_{j\bar{j}} = \partial_j g_{i\bar{j}} - \partial_j (g_0)_{i\bar{j}}.$$

Similarly $\partial_{\bar{i}} g_{j\bar{j}} = \partial_{\bar{j}} g_{i\bar{j}} - \partial_{\bar{j}} (g_0)_{i\bar{j}}$. Then (3.4) gives

$$\begin{aligned}
\frac{|\nabla \operatorname{tr}_{\omega_0} \omega|^2}{\operatorname{tr}_{\omega_0} \omega} &\leq \sum_{i,j} g^{i\bar{i}} g^{j\bar{j}} \partial_j g_{i\bar{j}} \partial_{\bar{j}} g_{j\bar{i}} - 2 \operatorname{Re} \left(\sum_{i,j} g^{i\bar{i}} g^{j\bar{j}} \partial_j g_{i\bar{j}} \partial_{\bar{j}} (g_0)_{j\bar{i}} \right) \\
&\quad + \sum_{i,j} g^{i\bar{i}} g^{j\bar{j}} \partial_j (g_0)_{i\bar{j}} \partial_{\bar{j}} (g_0)_{j\bar{i}} \\
&= \sum_{i,j} g^{i\bar{i}} g^{j\bar{j}} \partial_j g_{i\bar{j}} \partial_{\bar{j}} g_{j\bar{i}} - 2 \operatorname{Re} \left(\sum_{i,j} g^{i\bar{i}} g^{j\bar{j}} \partial_i g_{j\bar{j}} \partial_{\bar{j}} (g_0)_{j\bar{i}} \right) \\
&\quad - \sum_{i,j} g^{i\bar{i}} g^{j\bar{j}} \partial_j (g_0)_{i\bar{j}} \partial_{\bar{j}} (g_0)_{j\bar{i}} \\
&\leq \sum_{i,j} g^{i\bar{i}} g^{j\bar{j}} \partial_j g_{i\bar{j}} \partial_{\bar{j}} g_{j\bar{i}} - 2 \operatorname{Re} \left(\sum_{i \neq j} g^{i\bar{i}} g^{j\bar{j}} \partial_i g_{j\bar{j}} \partial_{\bar{j}} (g_0)_{j\bar{i}} \right) \\
&\leq \sum_{i,j} g^{i\bar{i}} g^{j\bar{j}} \partial_j g_{i\bar{j}} \partial_{\bar{j}} g_{j\bar{i}} + \sum_{i \neq j} g^{j\bar{j}} g^{j\bar{j}} |\partial_i g_{j\bar{j}}|^2 + \sum_{i \neq j} g^{i\bar{i}} g^{i\bar{i}} |\partial_j (g_0)_{i\bar{j}}|^2 \\
&\leq \sum_{i,j,k} g^{i\bar{i}} g^{j\bar{j}} \partial_k g_{i\bar{j}} \partial_{\bar{k}} g_{j\bar{i}} + \sum_{i,j} g^{i\bar{i}} g^{i\bar{i}} |\partial_j (g_0)_{i\bar{j}}|^2. \tag{3.5}
\end{aligned}$$

Thus we get the inequality (3.3). We will use it to prove our third order estimate.

Let $(\Gamma_0)_{ij}^k$ be the Christoffel symbols associated to g_0 . It is convenient to compute using $\Phi_{ij}^k = \Gamma_{ij}^k - (\Gamma_0)_{ij}^k$ as in [12] (see also [16]). Denote $|\cdot|$ the norm with respect

to the metric g . Consider

$$S = |\nabla_{g_0} g|^2 = g^{i\bar{p}} g^{j\bar{q}} g_{k\bar{r}} \Phi_{ij}{}^k \Phi_{\bar{p}\bar{q}}{}^{\bar{r}}.$$

Proposition 3.1. *There exist positive constants λ and C such that for $t \in [0, T)$,*

$$S \leq \frac{C}{|s|_h^{2\lambda}}.$$

Proof. Let

$$H = \frac{S}{(|s|_h^{-\alpha} - \text{tr}_{\omega_0} \omega)^2} + |s|_h^\beta \text{tr}_{\omega_0} \omega - At,$$

where α and β are constants to be determined and at least large enough such that

$$\frac{1}{2}|s|_h^{-\alpha} < |s|_h^{-\alpha} - \text{tr}_{\omega_0} \omega < |s|_h^{-\alpha}$$

and

$$|\nabla|s|_h^\beta| \leq C|s|_h^{\frac{3}{4}\beta}, \quad |\Delta|s|_h^\beta| \leq C|s|_h^{\frac{\beta}{2}}.$$

Computing the evolution of H ,

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta\right)H &= \frac{1}{(|s|_h^{-\alpha} - \text{tr}_{\omega_0} \omega)^2} \left(\frac{\partial}{\partial t} - \Delta\right)S \\ &\quad - \frac{2S}{(|s|_h^{-\alpha} - \text{tr}_{\omega_0} \omega)^3} \left(\frac{\partial}{\partial t} - \Delta\right)(|s|_h^{-\alpha} - \text{tr}_{\omega_0} \omega) \\ &\quad + \frac{4 \text{Re} \nabla S \cdot \bar{\nabla}(|s|_h^{-\alpha} - \text{tr}_{\omega_0} \omega)}{(|s|_h^{-\alpha} - \text{tr}_{\omega_0} \omega)^3} - \frac{6S|\nabla(|s|_h^{-\alpha} - \text{tr}_{\omega_0} \omega)|^2}{(|s|_h^{-\alpha} - \text{tr}_{\omega_0} \omega)^4} - A \\ &\quad + \left(\frac{\partial}{\partial t} - \Delta\right)(|s|_h^\beta \text{tr}_{\omega_0} \omega). \end{aligned} \tag{3.6}$$

From [16] and (3.1), we have the estimates

$$\left(\frac{\partial}{\partial t} - \Delta\right)S \leq C|s|_h^{-\alpha_1}(1 + S^{3/2}) - \frac{1}{2}|\bar{\nabla}\Phi|^2, \tag{3.7}$$

and

$$\left(\frac{\partial}{\partial t} - \Delta\right) \text{tr}_{\omega_0} \omega \leq -\frac{|s|_h^{\alpha_2} S}{C} + C|s|_h^{-\alpha_2} - \frac{1}{2} \sum_{i,j,k} g^{j\bar{j}} g^{i\bar{i}} \partial_k g_{i\bar{j}} \partial_{\bar{k}} g_{j\bar{i}}. \tag{3.8}$$

Also, by (3.3)

$$\frac{|\nabla \text{tr}_{\omega_0} \omega|^2}{\text{tr}_{\omega_0} \omega} \leq \sum_{i,j,k} g^{j\bar{j}} g^{i\bar{i}} \partial_k g_{i\bar{j}} \partial_{\bar{k}} g_{j\bar{i}} + C|s|_h^{-\alpha_3}.$$

Compute

$$\begin{aligned}
& \left(\frac{\partial}{\partial t} - \Delta\right)(|s|_h^\beta \operatorname{tr}_{\omega_0} \omega) \\
&= |s|_h^\beta \left(\frac{\partial}{\partial t} - \Delta\right) \operatorname{tr}_{\omega_0} \omega - (\operatorname{tr}_{\omega_0} \omega) \Delta |s|_h^\beta - 2 \operatorname{Re}(\nabla |s|_h^\beta \cdot \bar{\nabla} \operatorname{tr}_{\omega_0} \omega) \\
&\leq -\frac{1}{C'} |s|_h^{2\beta} S + C'.
\end{aligned}$$

In the last inequality we use

$$\begin{aligned}
2|\operatorname{Re}(\nabla |s|_h^\beta \cdot \bar{\nabla} \operatorname{tr}_{\omega_0} \omega)| &\leq C + \frac{1}{C} |\nabla |s|_h^\beta|^2 |\nabla \operatorname{tr}_{\omega_0} \omega|^2 \\
&\leq C + \frac{1}{C} |\nabla |s|_h^\beta|^2 (\operatorname{tr}_{\omega_0} \omega) \left(\sum_{i,j,k} g^{i\bar{i}} g^{j\bar{j}} \partial_k g_{i\bar{j}} \partial_{\bar{k}} g_{j\bar{i}} + C |s|_h^{-\alpha_3}\right) \\
&\leq C + \frac{1}{2} |s|_h^\beta \sum_{i,j,k} g^{i\bar{i}} g^{j\bar{j}} \partial_k g_{i\bar{j}} \partial_{\bar{k}} g_{j\bar{i}}.
\end{aligned}$$

for β large enough. Then fix β . Together with (3.7) and (3.8) we get

$$\begin{aligned}
\left(\frac{\partial}{\partial t} - \Delta\right) H &\leq C |s|_h^{2\alpha-\alpha_1} (1 + S^{3/2}) - \frac{1}{2} |s|_h^{2\alpha} |\bar{\nabla} \Phi|^2 \\
&\quad + C_0 |s|_h^{3\alpha} |\Delta |s|_h^{-\alpha}| S + C_0 |s|_h^{3\alpha-\alpha_2} S - |s|_h^{3\alpha+\alpha_2} \frac{S^2}{C_0} \\
&\quad + 4 \operatorname{Re} \frac{\nabla S \cdot \bar{\nabla} (|s|_h^{-\alpha} - \operatorname{tr}_{\omega_0} \omega)}{(|s|_h^{-\alpha} - \operatorname{tr}_{\omega_0} \omega)^3} + \left(-\frac{1}{C'} |s|_h^{2\beta} S + C'\right) - A.
\end{aligned}$$

As

$$|\bar{\nabla} S| \leq 2S^{1/2} |\bar{\nabla} \Phi|, \quad |\nabla \operatorname{tr}_{\omega_0} \omega| \leq C |s|_h^{-\alpha_4} S^{1/2}$$

and $|\nabla |s|_h^{-\alpha}| \leq C |s|_h^{-\alpha-\alpha_5}$, we have

$$\begin{aligned}
\left| 4 \operatorname{Re} \frac{\nabla S \cdot \bar{\nabla} (|s|_h^{-\alpha} - \operatorname{tr}_{\omega_0} \omega)}{(|s|_h^{-\alpha} - \operatorname{tr}_{\omega_0} \omega)^3} \right| &\leq C S^{1/2} |\bar{\nabla} \Phi| (|s|_h^{2\alpha-\alpha_5} + |s|_h^{3\alpha-\alpha_4} S^{1/2}) \\
&\leq \frac{1}{2} |s|_h^{2\alpha} |\bar{\nabla} \Phi|^2 + C_1 (|s|_h^{2\alpha-2\alpha_5} S + |s|_h^{4\alpha-2\alpha_4} S^2).
\end{aligned}$$

Also

$$C |s|_h^{2\alpha-\alpha_1} S^{3/2} \leq |s|_h^{3\alpha+\alpha_2} \frac{S^2}{2C_0} + C_1 |s|_h^{\alpha-2\alpha_1-\alpha_2} S.$$

By choosing α and then A sufficiently large, we have

$$\left(\frac{\partial}{\partial t} - \Delta\right) H \leq 0.$$

Thus H has a uniform upper bound and we obtain the desired estimate for S . \square

Proposition 3.2. *There exist positive constants λ and C such that for $t \in [0, T)$,*

$$|\text{Ric}| \leq \frac{C}{|s|_h^{2\lambda}}.$$

Proof. First, we have the evolution equation

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta\right)R_{j\bar{k}} = & \nabla_{\bar{k}}T_{lj}^r R_r^l + T_{lj}^r \nabla_{\bar{k}}R_r^l + R_{l\bar{k}j}^r R_r^l \\ & - R_{l\bar{k}}^{\bar{s}l} R_{j\bar{s}} + \nabla^{\bar{q}}T_{\bar{k}\bar{q}}^{\bar{s}} R_{j\bar{s}} + T_{\bar{k}\bar{q}}^{\bar{s}} \nabla^{\bar{q}}R_{j\bar{s}} \end{aligned}$$

Then it follows from Lemma 3.4 in [10] and Proposition 3.1 that

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta\right)|\text{Ric}| &= \frac{1}{2|\text{Ric}|} \left(\frac{\partial}{\partial t} - \Delta\right)|\text{Ric}|^2 + 2|\nabla|\text{Ric}||^2 \\ &\leq |s|_h^{-\alpha} (|\nabla|\text{Ric}|| + |\text{Rm}|^2 + 1) - \frac{|\nabla|\text{Ric}||^2}{|\text{Ric}|} + \frac{|\nabla|\text{Ric}||^2}{|\text{Ric}|}. \end{aligned} \quad (3.9)$$

for some constant $\alpha > 0$. Consider

$$H = |s|_h^{3\alpha} |\text{Ric}| + |s|_h^{4\beta} S - At,$$

where α and β are constants to be determined and are at least large enough such that

$$|\bar{\nabla}|s|_h^{3\alpha}| \leq C|s|_h^{2\alpha}, \quad |\Delta|s|_h^{3\alpha}| \leq C|s|_h^{2\alpha} \quad (3.10)$$

and

$$|\bar{\nabla}|s|_h^{4\beta}| \leq C|s|_h^{3\beta}, \quad |\Delta|s|_h^{4\beta}| \leq C|s|_h^{3\beta}. \quad (3.11)$$

Assume that H achieves maximum at a point (t_0, z_0) , $t_0 > 0$ and $|\text{Ric}| > 1$ at (t_0, z_0) . By (3.7) and the estimate for S ,

$$\left(\frac{\partial}{\partial t} - \Delta\right)S \leq C|s|_h^{-\beta} - \frac{1}{2}|\bar{\nabla}\Phi|^2, \quad (3.12)$$

for sufficiently large α . Together with (3.9), (3.10) and (3.11), we obtain

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta\right)H &\leq |s|_h^{2\alpha} (|\nabla|\text{Ric}|| + |\text{Rm}|^2) + |s|_h^{3\alpha} \left(\frac{|\nabla|\text{Ric}||^2}{|\text{Ric}|} - \frac{|\nabla|\text{Ric}||^2}{|\text{Ric}|} \right) \\ &\quad - 2\text{Re}(\nabla|\text{Ric}| \cdot \bar{\nabla}|s|_h^{3\alpha}) - \Delta|s|_h^{3\alpha} |\text{Ric}| - 2\text{Re}(\nabla S \cdot \bar{\nabla}|s|_h^{4\beta}) \\ &\quad + C|s|_h^{3\beta} - \frac{1}{2}|s|_h^{4\beta} |\bar{\nabla}\Phi|^2 - \Delta|s|_h^{4\beta} S - A. \end{aligned} \quad (3.13)$$

As $\nabla H = 0$ at (t_0, z_0) , we have

$$|s|_h^{3\alpha} \bar{\nabla}|\text{Ric}| = -\bar{\nabla}|s|_h^{3\alpha} |\text{Ric}| - |s|_h^{4\beta} \bar{\nabla}S - \bar{\nabla}|s|_h^{4\beta} S$$

Combining with (3.10) and (3.11) and $|\nabla|\text{Ric}|| \leq |\nabla\text{Ric}|$, we get

$$\begin{aligned} \frac{|\nabla|\text{Ric}||^2}{|s|_h^{-3\alpha}|\text{Ric}|} &\leq C|s|_h^{2\alpha}|\nabla\text{Ric}| + \frac{|\overline{\nabla}S||\nabla\text{Ric}|}{|s|_h^{-4\beta}|\text{Ric}|} + C\frac{S|\nabla\text{Ric}|}{|s|_h^{-3\beta}|\text{Ric}|} \\ &\leq \frac{|\nabla\text{Ric}|^2}{4|s|_h^{-3\alpha}|\text{Ric}|} + C_1|s|_h^\alpha|\text{Ric}| + C_2\frac{|s|_h^{7\beta}|\overline{\nabla}\Phi|^2}{|s|_h^{3\alpha}|\text{Ric}|} \\ &\quad + C_3\frac{|s|_h^{4\beta}}{|s|_h^{3\alpha}|\text{Ric}|}, \end{aligned} \tag{3.14}$$

where we use $|\overline{\nabla}S| \leq 2S|\overline{\nabla}\Phi|^2$ and $S \leq C|s|_h^{-\beta}$ in the second inequality. Note that

$$|\text{Rm}|^2 \leq \frac{3}{2}|\overline{\nabla}\Phi|^2 + |s|_h^{-2\alpha}$$

for α large as

$$\nabla_{\bar{q}}\Phi_{ij}{}^k = -R_{i\bar{q}j}{}^k + (R_0)_{i\bar{q}j}{}^k.$$

Then we have

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta\right)H &\leq C|s|_h^\alpha|\text{Ric}| + C\frac{|s|_h^{7\beta}|\overline{\nabla}\Phi|^2}{|s|_h^{3\alpha}|\text{Ric}|} + C\frac{|s|_h^{4\beta}}{|s|_h^{3\alpha}|\text{Ric}|} \\ &\quad + C|s|_h^{2\beta}|\overline{\nabla}\Phi| - \frac{1}{4}|s|_h^{4\beta}|\overline{\nabla}\Phi|^2 + C - A. \end{aligned}$$

Assume at (t_o, z_o) , $|s|_h^{3\alpha}|\text{Ric}| \geq 1$ (otherwise, $H \leq 2$ and the bound for $|\text{Ric}|$ follows). By (3.1),

$$|\text{Ric}| \leq |\overline{\nabla}\Phi| + |s|_h^{-\alpha_1}.$$

For α large enough, the term $|s|_h^\alpha|\text{Ric}|$ can be controlled by $\frac{1}{6}|s|_h^{4\beta}|\overline{\nabla}\Phi|^2$. Choosing α and then A sufficiently large, we get

$$\left(\frac{\partial}{\partial t} - \Delta\right)H \leq 0,$$

therefore we have the uniform upper bound for H and the proposition follows. \square

4. SMOOTH CONVERGENCE AS $t \rightarrow T^+$

In this section, we will prove sharper bounds on φ_ϵ and φ and then obtain the smooth convergence of the metrics $\omega(t)$ to $\tilde{\omega}_T$ on compact subsets of $N \setminus \{y_0\}$ as $t \rightarrow T^+$.

Recall that $\tilde{\omega}_{T,\epsilon} = \omega_N + \sqrt{-1}\partial\bar{\partial}\psi_{T,\epsilon}$. For simplicity, write $\tilde{\omega} = \tilde{\omega}_{T,\epsilon}$, $\hat{\omega} = \hat{\omega}_T = \omega_N$ and $|s|_h^2$ for $(\pi|_{M \setminus E}^{-1})^*(|s|_h^2)$ in the proof of the following lemma. Then

$$\tilde{\omega}^n = (\hat{\omega} + \sqrt{-1}\partial\bar{\partial}\psi_{T,\epsilon})^n = e^{F_\epsilon}\hat{\omega}^n, \quad \text{where } F_\epsilon = \log\left(\frac{C_\epsilon\Omega_\epsilon}{\hat{\omega}^n}\right).$$

Lemma 4.1. *There exist constants $\lambda > 0$ and $C > 0$, independent of ϵ , such that*

$$\frac{|s|_h^{2\lambda}}{C} \omega_N \leq \tilde{\omega}_{T,\epsilon} \leq \frac{C}{|s|_h^{2\lambda}} \omega_N$$

on $N \setminus \{y_0\}$.

Proof. As $\psi_{T,\epsilon}$ is uniformly bounded by (2.3), there exists a constant C_0 such that $\psi_{T,\epsilon} + C_0 \geq 1$. Take ϵ_0 small enough such that $\hat{\omega} - \epsilon_0 R_h \geq c\hat{\omega}$ for some positive constant c , where $R_h = -\sqrt{-1}\partial\bar{\partial} \log |s|_h^2$ is the curvature of the Hermitian metric h . Let $\tilde{\psi}_{T,\epsilon} = \psi_{T,\epsilon} - \epsilon_0 \log |s|_h^2$ and define

$$H = \log \operatorname{tr}_{\hat{\omega}} \tilde{\omega} - A\tilde{\psi}_{T,\epsilon} + \frac{1}{\tilde{\psi}_{T,\epsilon} + C_0}.$$

Note that $H(y)$ goes to negative infinity as y tends to y_0 . Compute at a point in $N \setminus \{y_0\}$. Assume $\operatorname{tr}_{\hat{\omega}} \tilde{\omega} \geq 1$ at this point. From section 9 of [20], we have

$$\Delta_{\tilde{\omega}} \log \operatorname{tr}_{\hat{\omega}} \tilde{\omega} \geq \frac{2}{(\operatorname{tr}_{\hat{\omega}} \tilde{\omega})^2} \operatorname{Re} \left(\tilde{g}^{k\bar{q}} \hat{T}_{p\bar{k}}^p \partial_{\bar{q}} \operatorname{tr}_{\hat{\omega}} \tilde{\omega} \right) - C \operatorname{tr}_{\hat{\omega}} \tilde{\omega} - |\Delta_{\tilde{\omega}} F_{\epsilon}| - C. \quad (4.1)$$

Assume that H achieves a maximum at z_0 . As $\partial_{\bar{q}} H = 0$ at z_0 , that is

$$\frac{\partial_{\bar{q}} \operatorname{tr}_{\hat{\omega}} \tilde{\omega}}{\operatorname{tr}_{\hat{\omega}} \tilde{\omega}} - A \partial_{\bar{q}} \tilde{\psi}_{T,\epsilon} - \frac{\partial_{\bar{q}} \tilde{\psi}_{T,\epsilon}}{(\tilde{\psi}_{T,\epsilon} + C_0)^2} = 0,$$

we get

$$\begin{aligned} & \left| \frac{2}{(\operatorname{tr}_{\hat{\omega}} \tilde{\omega})^2} \operatorname{Re}(\tilde{g}^{k\bar{q}} \hat{T}_{j\bar{k}}^j \hat{\nabla}_{\bar{q}} \operatorname{tr}_{\hat{\omega}} \tilde{\omega}) \right| \\ &= \left| \frac{2}{\operatorname{tr}_{\hat{\omega}} \tilde{\omega}} \operatorname{Re}(\tilde{g}^{k\bar{q}} \hat{T}_{j\bar{k}}^j (A + \frac{1}{(\tilde{\psi}_{T,\epsilon} + C_0)^2}) \partial_{\bar{q}} \tilde{\psi}_{T,\epsilon}) \right| \\ &\leq \frac{|\partial \tilde{\psi}_{T,\epsilon}|_{\tilde{\omega}}^2}{(\tilde{\psi}_{T,\epsilon} + C_0)^3} + \frac{(\tilde{\psi}_{T,\epsilon} + C_0)^3 C A^2 \operatorname{tr}_{\hat{\omega}} \tilde{\omega}}{(\operatorname{tr}_{\hat{\omega}} \tilde{\omega})^2}. \end{aligned} \quad (4.2)$$

If at z_0 , $(\operatorname{tr}_{\hat{\omega}} \tilde{\omega})^2 \leq A^2(\tilde{\psi}_{T,\epsilon} + C_0)^3$, then

$$H \leq \log A + \frac{3}{2} \log(\tilde{\psi}_{T,\epsilon} + C_0) - A\tilde{\psi}_{T,\epsilon} + \frac{1}{\tilde{\psi}_{T,\epsilon} + C_0}.$$

As $\tilde{\psi}_{T,\epsilon} + C_0 \geq 1$, we have an upper bound for H and thus $\tilde{\omega}$ is bounded from above. Otherwise, $A^2(\tilde{\psi}_{T,\epsilon} + C_0)^3 \leq (\operatorname{tr}_{\hat{\omega}} \tilde{\omega})^2$ at the maximum point. Moreover, by the definition of Ω_{ϵ} , it follows from (3.1) and Proposition 3.2 that

$$|\Delta_{\hat{\omega}} F_{\epsilon}| \leq \frac{C}{|s|^{2\beta}}$$

for uniform constants C and β . Together with (4.1) and (4.2) we have

$$\begin{aligned}
\Delta_{\tilde{\omega}} H &= \Delta_{\tilde{\omega}} \log \operatorname{tr}_{\tilde{\omega}} \tilde{\omega} + \frac{2|\partial \tilde{\psi}_{T,\epsilon}|_{\tilde{\omega}}^2}{(\tilde{\psi}_{T,\epsilon} + C_0)^3} \\
&\quad - \left(A + \frac{1}{(\tilde{\psi}_{T,\epsilon} + C_0)^2} \right) \operatorname{tr}_{\tilde{\omega}} (\tilde{\omega} - \hat{\omega} + \epsilon_0 R_h) \\
&\geq \Delta_{\tilde{\omega}} \log \operatorname{tr}_{\tilde{\omega}} \tilde{\omega} + cA \operatorname{tr}_{\tilde{\omega}} \hat{\omega} + \frac{2|\partial \tilde{\psi}_{T,\epsilon}|_{\tilde{\omega}}^2}{(\tilde{\psi}_{T,\epsilon} + C_0)^3} - (A+1)n \\
&\geq cA \operatorname{tr}_{\tilde{\omega}} \hat{\omega} - \frac{(\tilde{\psi}_{T,\epsilon} + C_0)^3 C A^2 \operatorname{tr}_{\tilde{\omega}} \hat{\omega}}{(\operatorname{tr}_{\tilde{\omega}} \tilde{\omega})^2} - \frac{C}{|s|_h^{2\lambda}} \\
&\geq (cA - C) \operatorname{tr}_{\tilde{\omega}} \hat{\omega} - \frac{C}{|s|_h^{2\lambda}}. \tag{4.3}
\end{aligned}$$

At the maximum point, $\Delta_{\tilde{\omega}} H \leq 0$, therefore

$$\operatorname{tr}_{\tilde{\omega}} \hat{\omega} \leq \frac{C}{|s|_h^{2\lambda}}$$

for sufficiently large A . Then at z_0 ,

$$\operatorname{tr}_{\tilde{\omega}} \tilde{\omega} \leq \frac{1}{(n-1)!} (\operatorname{tr}_{\tilde{\omega}} \hat{\omega})^{n-1} \frac{\tilde{\omega}^n}{\hat{\omega}^n} \leq \frac{C}{|s|_h^{2\beta}}$$

as $\tilde{\omega}^n = C_\epsilon \Omega_\epsilon$ has an upper bound by definition of Ω_ϵ and (3.1) holds. Thus there exists $C > 0$, independent of ϵ , such that $H \leq C$ for sufficiently large A . Since $\tilde{\psi}_{T,\epsilon} + C_0 \geq 1$, we obtain the desired estimates for $\tilde{\omega}_{T,\epsilon} = \tilde{\omega}$. \square

Recall that

$$\omega_\epsilon(t) = \hat{\omega}_t + \sqrt{-1} \partial \bar{\partial} \varphi_\epsilon \tag{4.4}$$

for $t \in [T, T']$. From Lemma 5.4 of [14], we have the following volume bound.

Lemma 4.2. *There exist constants $\lambda > 0$ and $C > 0$, independent of ϵ , such that*

$$\frac{\omega_\epsilon^n}{\Omega_N} \leq \frac{C}{|s|_h^{2\lambda}}$$

on $[T, T'] \times (N \setminus \{y_0\})$.

With Lemma 4.1 and Lemma 4.2, the upper bound for $\omega(t)$ can be obtained by using the argument of Tosatti-Weinkove [21] (see also [11]). For simplicity, we write $\hat{\omega} = \omega_N$, $\tilde{\omega} = \omega_\epsilon$ in the proof of the following lemma.

Lemma 4.3. *There exist constants $\lambda > 0$ and $C > 0$, independent of ϵ , such that on $[T, T'] \times (N \setminus \{y_0\})$,*

$$\frac{|s|_h^{2\lambda}}{C} \omega_N \leq \omega_\epsilon \leq \frac{C}{|s|_h^{2\lambda}} \omega_N.$$

Proof. Take ϵ_0 small enough such that $\hat{\omega}_t - \epsilon_0 R_h \geq c\hat{\omega}$ for any $t \in [T, T']$ for some constant $c > 0$. Let $\tilde{\varphi}_\epsilon = \varphi_\epsilon - \epsilon_0 \log |s|_h^2$. By Proposition 2.1, there exists a positive constant C_0 , such that $\tilde{\varphi}_\epsilon + C_0 \geq 1$. Define

$$H = \log \operatorname{tr}_{\tilde{\omega}} \tilde{\omega} - A\tilde{\varphi}_\epsilon + \frac{1}{\tilde{\varphi}_\epsilon + C_0},$$

where A is a positive constant to be determined. We have

$$H|_{t=T} \leq |s|_h^{2A\epsilon_0} \log \operatorname{tr}_{\omega_N} \tilde{\omega}_{T,\epsilon} - A\psi_{T,\epsilon} + 1.$$

By Lemma 4.1, H is uniformly bounded from above at time T . Moreover, $H(t, y)$ tends to negative infinity as y tends to y_0 , for any $t \in [T, T']$. Compute at a point in $N \setminus \{y_0\}$ with $\operatorname{tr}_{\tilde{\omega}} \tilde{\omega} \geq 1$. From Proposition 3.1 in [20],

$$\left(\frac{\partial}{\partial t} - \Delta_{\tilde{\omega}} \right) \log \operatorname{tr}_{\tilde{\omega}} \tilde{\omega} \leq \frac{2}{(\operatorname{tr}_{\tilde{\omega}} \tilde{\omega})^2} \operatorname{Re} \left(\tilde{g}^{k\bar{q}} \hat{T}_{kp}^p \partial_{\bar{q}} \operatorname{tr}_{\tilde{\omega}} \tilde{\omega} \right) + C \operatorname{tr}_{\tilde{\omega}} \hat{\omega}. \quad (4.5)$$

Assume H achieves its maximum at (t_0, z_0) , we have $\partial_{\bar{q}} \operatorname{tr}_{\tilde{\omega}} \tilde{\omega} = 0$ at this point, thus

$$\begin{aligned} & \left| \frac{2}{(\operatorname{tr}_{\tilde{\omega}} \tilde{\omega})^2} \operatorname{Re} \left(\tilde{g}^{k\bar{q}} \hat{T}_{kp}^p \partial_{\bar{q}} \operatorname{tr}_{\tilde{\omega}} \tilde{\omega} \right) \right| \\ & \leq \left| \frac{2}{\operatorname{tr}_{\tilde{\omega}} \tilde{\omega}} \operatorname{Re} \left(\tilde{g}^{k\bar{q}} \hat{T}_{kp}^p \left(A + \frac{1}{(\tilde{\varphi}_\epsilon + C_0)^2} \right) \partial_{\bar{q}} \tilde{\varphi}_\epsilon \right) \right| \\ & \leq \frac{|\partial \tilde{\varphi}_\epsilon|^2}{(\tilde{\varphi}_\epsilon + C_0)^3} + CA^2 (\tilde{\varphi}_\epsilon + C_0)^3 \frac{\operatorname{tr}_{\tilde{\omega}} \hat{\omega}}{(\operatorname{tr}_{\tilde{\omega}} \tilde{\omega})^2}. \end{aligned} \quad (4.6)$$

If at (t_0, z_0) , $(\operatorname{tr}_{\tilde{\omega}} \tilde{\omega})^2 \leq A^2 (\tilde{\varphi}_\epsilon + C_0)^3$, then

$$H \leq \log A + \frac{3}{2} \log (\tilde{\varphi}_\epsilon + C_0) - A\tilde{\varphi}_\epsilon + \frac{1}{\tilde{\varphi}_\epsilon + C_0}.$$

As $\tilde{\varphi}_\epsilon + C_0 \geq 1$, we have an upper bound for H and thus $\tilde{\omega}$ is bounded from above. Otherwise, $A^2 (\tilde{\varphi}_\epsilon + C_0)^3 \leq (\operatorname{tr}_{\tilde{\omega}} \tilde{\omega})^2$ at the maximum point. Computing the evolution

of H , it follows from (4.5) and (4.6) at (t_0, z_0)

$$\begin{aligned}
\left(\frac{\partial}{\partial t} - \Delta_{\tilde{\omega}}\right) H &\leq C \operatorname{tr}_{\tilde{\omega}} \hat{\omega} - \left(A + \frac{1}{\tilde{\varphi}_\epsilon + C_0}\right) \dot{\varphi}_\epsilon \\
&\quad + \left(A + \frac{1}{\tilde{\varphi}_\epsilon + C_0}\right) \operatorname{tr}_{\tilde{\omega}} (\tilde{\omega} - \hat{\omega}_t + \epsilon_0 R_h) \\
&\leq C \operatorname{tr}_{\tilde{\omega}} \hat{\omega} + (A+1) \log \frac{\Omega_N}{\tilde{\omega}^n} + C + (A+1)n \\
&\quad - A \operatorname{tr}_{\tilde{\omega}} (\hat{\omega}_t - \epsilon_0 R_h).
\end{aligned}$$

As $\hat{\omega}_t - \epsilon_0 R_h \geq c\hat{\omega}$ and $\left(\frac{\partial}{\partial t} - \Delta_{\tilde{\omega}}\right) H \leq 0$ at this point, we have

$$\operatorname{tr}_{\tilde{\omega}} \hat{\omega} \leq C \log \frac{\Omega_N}{\tilde{\omega}^n} + C_1.$$

for A large enough. Then at (t_0, z_0) ,

$$\begin{aligned}
\operatorname{tr}_{\tilde{\omega}} \tilde{\omega} &\leq \frac{1}{(n-1)!} (\operatorname{tr}_{\tilde{\omega}} \hat{\omega})^{n-1} \frac{\det \tilde{\omega}}{\det \hat{\omega}} \\
&\leq C \frac{\tilde{\omega}^n}{\Omega_N} \left(\log \frac{\Omega_N}{\tilde{\omega}^n}\right)^{n-1} + C' \\
&\leq \frac{C}{|s|_h^{2\beta}}
\end{aligned}$$

as $\tilde{\omega}^n/\Omega_N \leq \frac{C}{|s|_h^{2\lambda}}$ by Lemma 4.2. Hence there exists $C > 0$, independent of ϵ , such that $H \leq C$ for sufficiently large A . Since $\tilde{\varphi}_\epsilon + C_0 \geq 1$, we see that $\tilde{\omega}$ is uniformly bounded from above. The lower bound follows from an argument similar to the proof of Lemma 2.3 in [21]. \square

Proof of Theorem 1.2. The existence and uniqueness is given by Proposition 2.1. The characterization of the maximal time T_N follows from Theorem 1.2 in [20]. By Lemma 4.3, for any compact subset $K \subset N \setminus \{y_0\}$, there exists a positive constant C_K such that

$$\frac{\omega_N}{C_K} \leq \omega(t) \leq C_K \omega_N \quad \text{on } [T, T'] \times N.$$

The local estimates of Gill [5] then gives uniform C^∞ estimates for $\omega(t)$ on compact subsets of $N \setminus \{y_0\}$. The smooth convergence follows from this and we finish the proof. \square

5. PROOF OF THEOREM 1.3

In this section, we prove the smoothing property for the Chern-Ricci flow with rough initial data. We follow the arguments in Song-Tian [13] closely. Using the same notations as in the introduction, assume that ω_0 satisfies the condition (1.3),

then by the same arguments as in section 4 of [10], there exist functions $\psi_j \in PSH(M, \omega_0) \cap C^\infty(M)$ such that

$$\lim_{j \rightarrow \infty} \|\psi_j - \varphi_0\|_{L^\infty(M)} = 0. \quad (5.1)$$

Write $\omega_{0,j} = \omega_0 + \sqrt{-1}\partial\bar{\partial}\psi_j$. The Chern-Ricci flow starting at $\omega_{0,j}$ can be reduced to a parabolic complex Monge-Ampère equation. First denote

$$T = \sup\{t \geq 0 \mid \exists \psi \in C^\infty(M), \omega_0 - t \operatorname{Ric}(\omega_0) + \sqrt{-1}\partial\bar{\partial}\psi > 0\}.$$

Then for any $T' < T$, there exists $\psi_{T'} \in C^\infty(M)$ such that

$$\beta = \omega_0 - T' \operatorname{Ric}(\omega_0) + \sqrt{-1}\partial\bar{\partial}\psi_{T'} > 0.$$

Fix $T' < T$ and define smooth Hermitian metrics

$$\hat{\omega}_t = (1 - \frac{t}{T'})\omega_0 + \frac{t}{T'}\beta = \omega_0 + t\chi$$

on $[0, T']$, where $\chi = \frac{1}{T'}\sqrt{-1}\partial\bar{\partial}\psi_{T'} - \operatorname{Ric}(\omega_0)$. Let Ω be a volume form satisfying $\sqrt{-1}\partial\bar{\partial}\log \Omega = \frac{\partial}{\partial t}\hat{\omega}_t = \chi$. It follows that if φ_j solves the parabolic complex Monge-Ampère equation

$$\frac{\partial \varphi_j}{\partial t} = \log \frac{(\hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\varphi_j)^n}{\Omega}, \quad \varphi_j(0) = \psi_j \quad (5.2)$$

for $t \in [0, T']$, then $\omega_j = \hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\varphi_j$ solves the Chern-Ricci flow starting at $\omega_{0,j} = \omega_0 + \sqrt{-1}\partial\bar{\partial}\psi_j$. We will show uniform C^∞ bounds for φ_j and to prove Theorem 1.3.

Let

$$F = \frac{(\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_0)^n}{\Omega} \in L^p(M).$$

We use C, C', C_i, \dots to denote uniform constants depending only on $\omega_0, \|\varphi_0\|_{L^\infty(M)}$ and $\|F\|_{L^p(M)}$ and varying from line to line.

First we have the following two lemmas from [13] (Lemma 3.1 and 3.2). The proof is exactly the same as in [13].

Lemma 5.1. *There exists $C > 0$ such that for any $t \in [0, T']$,*

$$\|\varphi_j\|_{L^\infty(M)} \leq C.$$

Moreover, $\{\varphi_j\}$ is a Cauchy sequence in $C^0([0, T'] \times M)$, i.e.,

$$\lim_{j,k \rightarrow \infty} \|\varphi_j - \varphi_k\|_{L^\infty([0, T'] \times M)} = 0.$$

Lemma 5.2. *There exists $C > 0$ such that*

$$\frac{t^n}{C} \leq \frac{(\hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\varphi_j)^n}{\Omega} \leq e^{\frac{C}{t}}.$$

for any $t \in [0, T']$.

For convenience, we write $\omega' = \omega_j$ the solution of the Chern-Ricci flow starting at $\omega_{0,j}$ and $g', \Delta', |\cdot|_{g'}, \dots$ the notations corresponding to ω_j for a fix j . Similarly, we use ∇^0 to denote the covariant derivative with respect to g_0 . All the bounds obtained in the following lemmas are independent of j .

To prove the second order estimate, we will need the following proposition. It follows from Proposition 3.1 in [20] as $(T_0)_{ki\bar{l}} = (T_{0,j})_{ki\bar{l}}$, where T_0 and $T_{0,j}$ are the torsions corresponding to ω_0 and $\omega_{0,j}$.

Proposition 5.1. (Tosatti-Weinkove) *Assume that at a point $\text{tr}_{\omega_0} \omega' \geq 1$, then*

$$\left(\frac{\partial}{\partial t} - \Delta'\right) \log \text{tr}_{\omega_0} \omega' \leq \frac{2}{(\text{tr}_{\omega_0} \omega')^2} \text{Re}(g'^{p\bar{q}}(T_0)_{pi}^i \nabla_{\bar{q}}^0 \text{tr}_{\omega_0} \omega') + C \text{tr}_{\omega'} \omega_0$$

at this point for some constant C depending only on g_0 .

Lemma 5.3. *There exists $C > 0$ such that for $t \in (0, T]$,*

$$\text{tr}_{\omega_0} \omega' \leq e^{\frac{C}{t}}.$$

Proof. Let

$$H = t \log \text{tr}_{\omega_0} \omega' + e^\Psi,$$

where $\Psi = A(\sup_{[0, T'] \times M} \varphi_j - \varphi_j)$ and A is a constant to be chosen later. Assume that

H achieves its maximum at (t_0, z_0) and $\text{tr}_{\omega_0} \omega' > 1$ (otherwise we obtain the upper bound for $\text{tr}_{\omega_0} \omega'$ directly). Choose coordinates around (t_0, z_0) such that at this point, $(g_0)_{i\bar{j}} = \delta_{ij}$ and $(g'_{i\bar{j}})$ is diagonal. First we have

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta'\right) H &= t \left(\frac{\partial}{\partial t} - \Delta'\right) \log \text{tr}_{\omega_0} \omega' + \log \text{tr}_{\omega_0} \omega' - A e^\Psi \dot{\varphi}_j \\ &\quad + A e^\Psi \Delta' \varphi_j - A^2 e^\Psi |\nabla \varphi_j|_{g'}^2. \end{aligned}$$

It follows from Proposition 5.1 that

$$\left(\frac{\partial}{\partial t} - \Delta'\right) \log \text{tr}_{\omega_0} \omega' \leq \frac{2}{(\text{tr}_{\omega_0} \omega')^2} \text{Re}(g'^{k\bar{k}}(T_0)_{ki}^i \partial_{\bar{k}} \text{tr}_{\omega_0} \omega') + C \text{tr}_{\omega'} \omega_0.$$

At (t_0, z_0) , $\nabla_{\bar{k}} H = 0$ gives

$$t \frac{\partial_{\bar{k}} \text{tr}_{\omega_0} \omega'}{\text{tr}_{\omega_0} \omega'} - A e^\Psi \partial_{\bar{k}} \varphi_j = 0.$$

Then

$$\begin{aligned} \frac{2t}{(\text{tr}_{\omega_0} \omega')^2} \text{Re}(g'^{k\bar{k}}(T_0)_{ki}^i \partial_{\bar{k}} \text{tr}_{\omega_0} \omega') &\leq \frac{2A e^\Psi}{\text{tr}_{\omega_0} \omega'} |\text{Re}(g'^{k\bar{k}}(T_0)_{ki}^i \partial_{\bar{k}} \varphi_j)| \\ &\leq e^\Psi (A^2 |\nabla \varphi_j|_{g'}^2 + C_1 \text{tr}_{\omega'} \omega_0). \end{aligned} \tag{5.3}$$

Also

$$\text{tr}_{\omega_0} \omega' \leq \frac{1}{(n-1)!} (\text{tr}_{\omega'} \omega_0)^{n-1} \frac{(\omega')^n}{\omega_0^n}.$$

Combining all the above inequalities, we get

$$\begin{aligned}
& \left(\frac{\partial}{\partial t} - \Delta'\right)H \\
& \leq C_1 e^\Psi \operatorname{tr}_{\omega'} \omega_0 + Ct \operatorname{tr}_{\omega'} \omega_0 + \log \operatorname{tr}_{\omega_0} \omega' + Ane^\Psi - Ae^\Psi \operatorname{tr}_{\omega'} \hat{\omega}_t - Ae^\Psi \dot{\varphi}_j \\
& \leq -e^\Psi \operatorname{tr}_{\omega'} (A\hat{\omega}_t - C_1\omega_0 - Ct\omega_0) + (1 - Ae^\Psi) \log \frac{(\omega')^n}{\omega_0^n} + (n-1) \log \operatorname{tr}_{\omega'} \omega_0 + C_2 \\
& \leq -C \operatorname{tr}_{\omega'} \omega_0 - C_3 \log t + C_4.
\end{aligned} \tag{5.4}$$

If we choose A to be large enough. Then at (t_0, z_0) ,

$$C_5 \left(\frac{\omega_0^n}{(\omega')^n} \right)^{\frac{1}{n-1}} (\operatorname{tr}_{\omega_0} \omega')^{\frac{1}{n-1}} \leq C \operatorname{tr}_{\omega'} \omega_0 \leq -C_3 \log t + C_4.$$

So

$$\log \operatorname{tr}_{\omega_0} \omega' \leq \log \left(\left(\log \frac{1}{t} \right)^{n-1} \left(\frac{(\omega')^n}{\omega_0^n} \right) \right) + C_6 \leq \frac{C_7}{t} + C_8.$$

Thus H is uniformly bounded for $t \in (0, T]$ and we obtain the required estimate. \square

Now let $S = |\nabla_{g_0} g'|_{g'}^2$. For convenience we still denote $\Phi_{ij}{}^k = \Gamma'_{ij}{}^k - (\Gamma_0)_{ij}{}^k$, then

$$S = |\Phi|_{g'}^2 = g'^{i\bar{p}} g'^{j\bar{q}} g'_{k\bar{r}} \Phi_{ij}{}^k \Phi_{\bar{p}\bar{q}}{}^{\bar{r}}.$$

Lemma 5.4. *There exists $C > 0$ and $\lambda > 0$ such that for $t \in (0, T']$,*

$$S \leq C e^{\frac{\lambda}{t}}.$$

Proof. By the evolution equation for $\operatorname{tr}_{\omega_0} \omega'$ in [20] and Lemma 5.3, we have

$$\left(\frac{\partial}{\partial t} - \Delta'\right) \operatorname{tr}_{\omega_0} \omega' \leq -C_1 e^{-\frac{\alpha}{t}} S + C_2 e^{\frac{\alpha}{t}}$$

and

$$\left(\frac{\partial}{\partial t} - \Delta'\right) S \leq e^{\frac{\beta}{t}} (S^{3/2} + 1) - \frac{1}{2} (|\bar{\nabla}' \Phi|_{g'}^2 + |\nabla' \Phi|_{g'}^2).$$

for some positive constants α and β . Take $\lambda_1 > 0$ such that $\frac{1}{2} e^{\frac{\lambda_1}{t}} < e^{\frac{\lambda_1}{t}} - \operatorname{tr}_{\omega_0} \omega' < e^{\frac{\lambda_1}{t}}$. Let

$$H = \frac{S}{(e^{\frac{\lambda_1}{t}} - \operatorname{tr}_{\omega_0} \omega')^2} + e^{-\frac{\lambda_2}{t}} \operatorname{tr}_{\omega_0} \omega'.$$

Compute

$$\begin{aligned}
\left(\frac{\partial}{\partial t} - \Delta'\right)H &= \frac{1}{(e^{\frac{\lambda_1}{t}} - \text{tr}_{\omega_0} \omega')^2} \left(\frac{\partial}{\partial t} - \Delta'\right)S + \frac{2S}{(e^{\frac{\lambda_1}{t}} - \text{tr}_{\omega_0} \omega')^3} \left(\frac{\partial}{\partial t} - \Delta'\right) \text{tr}_{\omega_0} \omega' \\
&\quad - \frac{4 \text{Re } \nabla' \text{tr}_{\omega_0} \omega' \cdot \bar{\nabla}' S}{(e^{\frac{\lambda_1}{t}} - \text{tr}_{\omega_0} \omega')^3} - \frac{6S |\nabla' \text{tr}_{\omega_0} \omega'|^2}{(e^{\frac{\lambda_1}{t}} - \text{tr}_{\omega_0} \omega')^4} + \frac{\frac{2\lambda_1}{t^2} e^{\frac{\lambda_1}{t}} S}{(e^{\frac{\lambda_1}{t}} - \text{tr}_{\omega_0} \omega')^3} \\
&\quad + e^{-\frac{\lambda_2}{t}} \left(\frac{\partial}{\partial t} - \Delta'\right) \text{tr}_{\omega_0} \omega' + \frac{\lambda_2}{t^2} e^{-\frac{\lambda_2}{t}} \text{tr}_{\omega_0} \omega' \\
&\leq 4e^{-\frac{2\lambda_1}{t}} e^{\frac{\beta}{t}} (S^{3/2} + 1) - \frac{1}{2} e^{-\frac{2\lambda_1}{t}} (|\bar{\nabla}' \Phi|_{g'}^2 + |\nabla' \Phi|_{g'}^2) \\
&\quad + \left(-2C_1 e^{-\frac{\alpha}{t}} e^{-\frac{3\lambda_1}{t}} S^2 + 16C_2 e^{\frac{\alpha}{t}} e^{-\frac{3\lambda_1}{t}} S\right) + 32e^{-\frac{3\lambda_1}{t}} |\text{Re } \nabla' \text{tr}_{\omega_0} \omega' \cdot \bar{\nabla}' S| \\
&\quad + C_3 e^{-\frac{\lambda_1}{t}} S + \left(-C_1 e^{-\frac{\alpha}{t}} e^{-\frac{\lambda_2}{t}} S + C_2 e^{\frac{\alpha}{t}} e^{-\frac{\lambda_2}{t}}\right) + C_4
\end{aligned}$$

where we choose λ_2 large enough such that $2\frac{\lambda_2}{t^2} e^{-\frac{\lambda_2}{t}} \text{tr}_{\omega_0} \omega' \leq C_4$ for some constant C_4 . Note that $|\nabla' \text{tr}_{\omega_0} \omega'|_{g'} \leq \frac{1}{64} e^{\frac{\gamma}{t}} S^{1/2}$ for some $\gamma > 0$ and $|\bar{\nabla}' S|_{g'} \leq 2S^{1/2} |\bar{\nabla}' \Phi|_{g'}$, we have

$$\begin{aligned}
32e^{-\frac{3\lambda_1}{t}} |\text{Re } \nabla' \text{tr}_{\omega_0} \omega' \cdot \bar{\nabla}' S| &\leq e^{-\frac{3\lambda_1}{t}} e^{\frac{\gamma}{t}} S |\bar{\nabla}' \Phi|_{g'} \\
&\leq \frac{1}{2} e^{-\frac{2\lambda_1}{t}} |\bar{\nabla}' \Phi|_{g'}^2 + \frac{1}{2} e^{-\frac{4\lambda_1}{t}} e^{\frac{2\gamma}{t}} S^2.
\end{aligned}$$

Also we have

$$4e^{-\frac{2\lambda_1}{t}} e^{\frac{\beta}{t}} S^{3/2} \leq C_1 e^{-\frac{\alpha}{t}} e^{-\frac{3\lambda_1}{t}} S^2 + \frac{4}{C_1} e^{\frac{\alpha}{t}} e^{\frac{2\beta}{t}} e^{-\frac{\lambda_1}{t}} S.$$

Take λ_2 sufficiently large such that $e^{\frac{\alpha}{t}} e^{-\frac{\lambda_2}{t}} < 1$, then fix λ_2 . Let $\lambda_1 \geq \alpha + \lambda_2$ be large enough such that

$$\left(\frac{\partial}{\partial t} - \Delta'\right)H \leq -\frac{1}{2} C_1 e^{-\frac{\alpha}{t}} e^{-\frac{\lambda_2}{t}} S + C$$

for some constant C . Assume that H achieves its maximal at (t_0, z_0) , $t_0 > 0$, then at this point

$$0 \leq -\frac{1}{2} C_1 e^{-\frac{\alpha}{t}} e^{-\frac{\lambda_2}{t}} S + C.$$

It follows that H is bounded by some constant. Therefore $S \leq C e^{\frac{\lambda}{t}}$ for some constants $C > 0$ and $\lambda > 0$. \square

In addition, to bound the derivatives of ω_j in the t -direction, it is sufficient to bound $|\text{Ric}(g)|$ which follows from the proof of Lemma 3.4 in [10]. Then by the standard parabolic estimates [9], we obtain all the higher order estimates.

Proposition 5.2. *For any $0 < \epsilon < T'$ and $k \geq 0$, there exists $C_{\epsilon, T', k} > 0$, such that*

$$\|\varphi_j\|_{C^k([\epsilon, T'] \times M)} \leq C_{\epsilon, T', k}.$$

The proposition below follows from the same arguments for the proof of Proposition 3.3 in [13]. For reader's convenience, we provide the proof here.

Proposition 5.3. *There exists a function $\varphi \in C^0([0, T] \times M) \cap C^\infty((0, T) \times M)$ such that φ is the unique solution of the equation*

$$\frac{\partial \varphi}{\partial t} = \log \frac{(\hat{\omega}_t + \sqrt{-1} \partial \bar{\partial} \varphi)^n}{\Omega}, \quad \text{for } t \in (0, T), \quad \varphi|_{t=0} = \varphi_0. \quad (5.5)$$

Proof. By Lemma 5.1, φ_j is a Cauchy sequence in $C^0([0, T'] \times M)$ and so we can define $\varphi = \lim_{j \rightarrow \infty} \varphi_j$ which is in $C^0([0, T'] \times M)$. Then it follows from Proposition 5.2 that for any $0 < \epsilon < T' < T$, φ_j converges to φ in $C^\infty([\epsilon, T'] \times M)$. Therefore $\varphi \in C^\infty((0, T) \times M)$ satisfying the above equation on $(0, T)$. Note that

$$\lim_{t \rightarrow 0^+} \|\varphi(t, \cdot) - \varphi_0(\cdot)\|_{L^\infty(M)} = 0$$

as $\varphi_j(0) = \psi_j$ and $\psi_j \rightarrow \varphi_0$ in $L^\infty(M)$ as $j \rightarrow \infty$. Then $\varphi|_{t=0} = \varphi_0$ and we have the existence of a solution $\varphi \in C^0([0, T] \times M) \cap C^\infty((0, T) \times M)$ for equation (5.5). To prove the uniqueness, we assume that there is another solution $\tilde{\varphi} \in C^0([0, T] \times M) \cap C^\infty((0, T) \times M)$ of equation (5.5). Let $\psi = \tilde{\varphi} - \varphi$. Then ψ solves the equation

$$\frac{\partial \psi}{\partial t} = \log \frac{(\hat{\omega}_t + \sqrt{-1} \partial \bar{\partial} \varphi + \sqrt{-1} \partial \bar{\partial} \psi)^n}{(\hat{\omega}_t + \sqrt{-1} \partial \bar{\partial} \varphi)^n}, \quad \text{for } t \in (0, T), \quad \psi|_{t=0} = 0.$$

At any given time t , the maximum of ψ is achieved at some point $z \in M$, then $\frac{d\psi_{\max}(t)}{dt} \leq 0$ a.e. in $[0, T]$. Similarly we have $\frac{d\psi_{\min}(t)}{dt} \geq 0$ a.e. in $[0, T]$. As both $\psi_{\max}(t)$ and $\psi_{\min}(t)$ are absolutely continuous on $[0, T]$ with $\psi_{\max}(0) = \psi_{\min}(0) = 0$, we have $\psi_{\max}(t) \leq 0 \leq \psi_{\min}(t)$ for $t \in [0, T]$. Hence $\psi(t) = 0$ for $t \in [0, T]$. \square

Now we can prove the smoothing property for the Chern-Ricci flow with rough initial data.

Proof of Theorem 1.3. If φ is a solution of (5.5), then taking $\sqrt{-1} \partial \bar{\partial}$ of (5.5) shows that $\omega = \hat{\omega}_t + \sqrt{-1} \partial \bar{\partial} \varphi$ solves the Chern-Ricci flow on $(0, T)$ and $\lim_{t \rightarrow 0^+} \|\varphi(t, \cdot) - \varphi_0(\cdot)\|_{L^\infty(M)} = 0$. Conversely, if ω solves (1.1), then

$$\frac{\partial}{\partial t}(\omega - \hat{\omega}_t) = \sqrt{-1} \partial \bar{\partial} \log \frac{\omega^n}{\Omega}.$$

Thus $\omega(t)$ must be of the form $\omega(t) = \hat{\omega}_t + \sqrt{-1} \partial \bar{\partial} \varphi$ for some φ solving the equation

$$\sqrt{-1} \partial \bar{\partial} \left(\frac{\partial \varphi}{\partial t} - \log \frac{(\hat{\omega}_t + \sqrt{-1} \partial \bar{\partial} \varphi)^n}{\Omega} \right) = 0. \quad (5.6)$$

Proposition 5.3 gives a solution of the above equation. Suppose that there exists another solution $\tilde{\varphi} \in C^\infty((0, T) \times M) \cap C^0([0, T] \times M)$ of equation (5.6). Then

$$\frac{\partial \tilde{\varphi}}{\partial t} = \log \frac{(\hat{\omega}_t + \sqrt{-1} \partial \bar{\partial} \tilde{\varphi})^n}{\Omega} + f(t)$$

with $\lim_{t \rightarrow 0^+} \tilde{\varphi}(t) = \varphi_0$ for some smooth function $f(t)$. So we get $\varphi = \tilde{\varphi} - \int_0^t f(s)ds$ is a solution of the equation (5.5) which is unique. Therefore $\tilde{\varphi} = \varphi + \int_0^t f(s)ds$ and we prove the uniqueness. \square

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